Zeros of the Riemann Zeta-Function and Hardy's Theorem

The Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} , \quad \mathrm{Re}\, s = \sigma > 1 .$$

Euler product:

$$\zeta(s) = \prod_p (1-p^{-s})^{-1}.$$

Euler (1744): $\lim_{s\to 1^+} \zeta(s) = +\infty \iff \sum_p \frac{1}{p}$ diverges.

Euler (1755): $\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n} B_{2n}}{2\cdot(2n)!} , \quad n \geq 1 ,$

B_n is the n-th Bernoulli number.

Riemann (1860): 1. $\zeta(s)$ can be countinued to a meromorphic function in the complex plane with a simple pole at $s=1$ with residue 1.

2. Fuctional equation:

$$\pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma(\tfrac{1-s}{2}) \zeta(1-s) .$$

$\Rightarrow$ zeros with $\sigma < 0$: $-2, -4, \cdots, -2n, \cdots$ (trivial zeros)

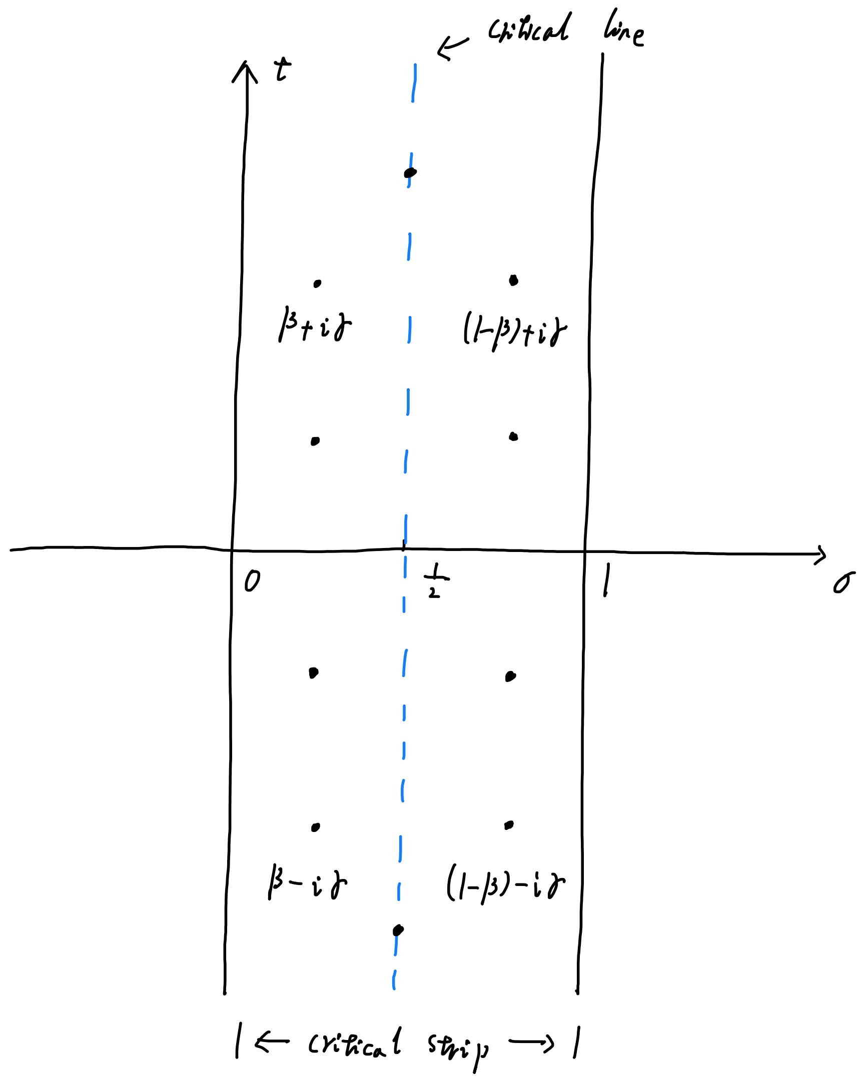
Hadamard (1893): The function

$$\xi(s) := \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \zeta(s)$$

is entire of order 1 with Hadamard product

$$\xi(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}.$$

$\Rightarrow$ $\zeta(s)$ has infinitely many zeros $\rho = \beta + i\gamma$ with $0 \leq \beta \leq 1$ (called nontrivial zeros).



$$N(T) := \# \{ \rho = \beta + i\gamma : \quad 0 < \gamma \leq T \}.$$

Von Mangoldt (1905):

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \quad \text{as} \quad T \to \infty.$$

Arranging the zeros so that $0 < \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n \leq \cdots$, we have $\gamma_n \sim \frac{2\pi n}{\log n}$.

Littlewood (1924): $\quad \gamma_{n+1} - \gamma_n \to 0$.

Riemann Hypothesis: All the nontrivial zeros lie on the critical line $\sigma = \frac{1}{2}$!

Why important?

Example 1. (The Distribution of Primes). Let

$$\pi(x) := \#\{p \leq x : p \text{ is prime}\}.$$

PNT: $\pi(x) = li(x) + E(x)$ with $E(x) = o(li(x))$.

$$li(x) = \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}, \qquad E(x) \text{ error}$$

Current methods $\Rightarrow$ $E(x) = O(x e^{-c(\log x)^{\frac{3}{5}}/(\log\log x)^{\frac{1}{5}}})$, $c = 0.2098$ (Ford 2002).

RH $\Leftrightarrow$ $E(x) = O(x^{\frac{1}{2}} \log x)$.

Example 2. (The Distribution of Primes in Short Intervals) Note that

$$\text{PNT} \Rightarrow \pi(x+y) - \pi(x) \sim \frac{y}{\log x} \quad \text{if} \quad x e^{-c(\log x)^{\frac{3}{5}}/(\log\log x)^{\frac{1}{5}}} \leq y \leq x.$$

How small can y be compared to x so that the above asymptotic still holds?

Zero density estimates $\Rightarrow$ $x^{\frac{7}{12}+\varepsilon} \leq y \leq x$, $\forall \varepsilon > 0$ (Huxley 1972)

RH $\Rightarrow$ $x^{\frac{1}{2}} \log x \leq y \leq x$.

Prime gaps: Sieve methods $\Rightarrow$ $p_{n+1} - p_n = O(p_n^{0.535})$ (Baker & Harmann 1996)

RH $\Rightarrow$ $p_{n+1} - p_n = O(p_n^{\frac{1}{2}} \log p_n)$

Maybe $p_{n+1} - p_n = O((\log p_n)^2)$?

Example 3. (The Growth of $\zeta(1+it)$ and $\zeta(\frac{1}{2}+it)$ for large $|t|$)

$$\text{Current methods} \Rightarrow \zeta(1+it) = O\left((\log|t|)^{\frac{2}{3}}\right) \quad (\text{Richert } 1967),$$
$$RH \Rightarrow \zeta(1+it) = O(\log\log|t|).$$

$$\text{Current methods} \Rightarrow \zeta(\tfrac{1}{2}+it) = O(|t|^{\alpha}) \quad \text{for various numerical values of } \alpha > 0,$$
$$RH \Rightarrow \zeta(\tfrac{1}{2}+it) = O(|t|^{\varepsilon}), \quad \forall\, \varepsilon > 0 \quad (\text{Lindelöf hypothesis}).$$

Example 4. (The Sum of Divisors) Let $\sigma(n)$ denote the sum of all the positive divisors of $n \geq 1$. We seek upper bounds for $\sigma(n)$.

$$\text{Current methods} \Rightarrow \frac{\sigma(n)}{n} < e^{\gamma}\log\log n + \frac{2/3}{\log\log n}, \quad n \geq 3 \quad (\text{Robin } 1984)$$
$$RH \Leftrightarrow \frac{\sigma(n)}{n} < e^{\gamma}\log\log n, \quad n \geq 5041 \quad (\text{The Ramanujan–Robin thm}).$$

Unconditional results on zeros of $\zeta(s)$ on the critical line

Let

$$N_0(T) := \#\{\rho = \tfrac{1}{2} + i\gamma : 0 < \gamma \leq T\}.$$

Recall

$$N(T) = \#\{\rho = \beta + i\gamma : 0 < \gamma \leq T\}.$$

So $RH \Leftrightarrow N_0(T) = N(T)$.

Hardy (1914): $N_0(T) \to \infty$ as $T \to \infty$.

Hardy & Littlewood (1920): $N_0(T) > AT$.

Selberg (1942): $N_0(T) > A\,N(T)$ $\Rightarrow$ a positive proportion of zeros lie on $\sigma = \frac{1}{2}$.

Levinson (1974): $N_0(T) > \frac{1}{3} N(T)$ $\Rightarrow$ $> 33\%$.

Conrey (1989): $N_0(T) > \frac{2}{5} N(T)$ $\Rightarrow$ $> 40\%$.

Remark.

Distribution of values of $\zeta(s)$ on $\sigma = \frac{1}{2}$: Selberg's central limit thm:

$$\frac{\log \zeta(\frac{1}{2}+it)}{\sqrt{\frac{1}{2}\log\log t}} \approx X + iY,$$

$X, Y \sim N(0,1)$ independent. The result for real part

$$\frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log t}} \approx X$$

can be proved by computing moments (Radziwiłł & Soundararajan 2017).

Proof that $N_0(T) \to \infty$

Conventions & Notation

1. Always write $s = \sigma + it$ and $z = x + iy$.
2. $\eta(t) := \xi(\frac{1}{2}+it)$

$$\left.\begin{aligned}\overline{\xi(s)} &= \xi(\bar{s}) \\ \xi(s) &= \xi(1-s)\end{aligned}\right\} \Rightarrow \eta(t) \text{ is real-valued and even.}$$

3. Jacobi theta function $\theta(z) := \sum_{n=-\infty}^{+\infty} e^{-n^2\pi z}$, $x > 0$.

Functional equation: $\theta(z^{-1}) = \sqrt{z}\,\theta(z)$.

Fact: $\lim\limits_{z \to i} \theta^{(k)}(z) = 0$, $\forall k \geq 0$.

Step 1. The Cohen-Mellin integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s) z^{-s}\,ds = e^{-z}, \quad \Re > 0, \; c > 0.$$

Mellin inversion of $\Gamma(s)$. We use this to derive a representation of $\theta(z)$ in terms of $\eta(t)$.

$$\theta(z) = \sum_{n=-\infty}^{+\infty} e^{-n^2\pi z} = 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi z} = 1 + 2\sum_{n=1}^{\infty} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)(n^2\pi z)^{-s}\,ds$$

$$= 1 + \frac{2}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\pi z)^{-s}\Gamma(s)\zeta(2s)\,ds, \quad c > \frac{1}{2}.$$

↑ check

Move the line of integration to $\sigma = \frac{1}{4}$:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\pi z)^{-s}\Gamma(s)\zeta(2s)\,ds = \frac{1}{2\pi i}\int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} (\pi z)^{-s}\Gamma(s)\zeta(2s)\,ds + \underbrace{\mathrm{Res}\left((\pi z)^{-s}\Gamma(s)\zeta(2s),\; s = \tfrac{1}{2}\right)}_{\frac{1}{2}(\pi z)^{-\frac{1}{2}}\underbrace{\Gamma(\frac{1}{2})}_{\sqrt{\pi}} = \frac{1}{2} z^{-\frac{1}{2}}}$$

$$\Rightarrow \quad \theta(z) = 1 + z^{-\frac{1}{2}} + \frac{1}{\pi i}\int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} (\pi z)^{-s}\Gamma(s)\zeta(2s)\,ds.$$

Recall $\xi(s) = \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$.

$$\theta(z) = 1 + z^{-\frac{1}{2}} + \frac{1}{\pi i} \int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} z^{-s} \cdot \frac{\xi(2s)}{s(2s-1)} \, ds$$

$$= 1 + z^{-\frac{1}{2}} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} z^{-\frac{1}{4} - i\frac{t}{2}} \frac{\xi(\frac{1}{2}+it)}{(\frac{1}{4}+i\frac{t}{2})(-\frac{1}{2}+it)} \, dt \quad \left(\text{Substituting } s = \frac{1}{4} + \frac{1}{2} it \right)$$

$$= 1 + z^{-\frac{1}{2}} - \frac{1}{\pi} \int_{-\infty}^{+\infty} z^{-\frac{1}{4} - i\frac{t}{2}} \frac{\eta(t)}{\frac{1}{4} + t^2} \, dt .$$

$$= 1 + z^{-\frac{1}{2}} - \frac{1}{\pi} \int_0^{+\infty} z^{-\frac{1}{4}} \left(z^{-i \cdot \frac{t}{2}} + z^{i \cdot \frac{t}{2}} \right) \cdot \frac{\eta(t)}{\frac{1}{4} + t^2} \, dt \quad (\eta(t) \text{ is even})$$

$$= 1 + z^{-\frac{1}{2}} - \frac{2}{\pi} \int_0^{\infty} z^{-\frac{1}{4}} \cos\left(\frac{t}{2} \log z \right) \cdot \frac{\eta(t)}{\frac{1}{4} + t^2} \, dt ,$$

valid for z with $x > 0$.

$\boxed{\text{Step 2.}}$ Evaluate for a near $\frac{\pi}{4}$

$$I_n(a) := \int_0^{\infty} t^{2n} \cosh(at) \cdot \frac{\eta(t)}{\frac{1}{4} + t^2} \, dt \quad (\cosh(w) = \cos(iw)).$$

① $|a| < \frac{\pi}{4}$, $\theta(e^{i \cdot 2a}) = \underbrace{1 + e^{-ia}}_{2e^{-i \cdot \frac{a}{2}} \cos \frac{a}{2}} - \underbrace{\frac{2}{\pi} \int_0^{\infty} e^{-i \cdot \frac{a}{2}} \cosh(at) \cdot \frac{\eta(t)}{\frac{1}{4} + t^2} \, dt}_{e^{-i \cdot \frac{a}{2}} I_0(a)}$

$$\Rightarrow I_0(a) = \pi \cos \frac{a}{2} - \frac{\pi}{2} e^{i \cdot \frac{a}{2}} \theta(e^{i \cdot 2a}).$$

② $I_n(a) = \int_0^\infty t^{2n} \cosh(at) \cdot \frac{\eta(t)}{\frac{1}{4}+t^2}\, dt = \int_0^\infty \left(\frac{d}{da}\right)^{2n} (\cosh(at)) \cdot \frac{\eta(t)}{\frac{1}{4}+t^2}\, dt$

$$\underset{\substack{\uparrow \\ \text{check}}}{=} \left(\frac{d}{da}\right)^{2n} \underbrace{\int_0^\infty \cosh(at) \cdot \frac{\eta(t)}{\frac{1}{4}+t^2}\, dt}_{I_0(a)}$$

$\boxed{I_0(a) = \pi \cos\frac{a}{2} - \frac{\pi}{2} e^{i\cdot\frac{a}{2}} \theta(e^{i\cdot 2a})}$

$$= \frac{(-1)^n}{2^{2n}} \pi \cos\frac{a}{2} - \frac{\pi}{2} \left(\frac{d}{da}\right)^{2n} \left(e^{i\cdot\frac{a}{2}} \theta(e^{i\cdot 2a})\right)$$

Leibniz rule + $\boxed{\lim_{z\to i} \theta^{(k)}(z) = 0}$ $\Rightarrow \lim_{a\to\frac{\pi}{4}} \left(\frac{d}{da}\right)^{2n} \left(e^{i\cdot\frac{a}{2}} \theta(e^{i\cdot 2a})\right) = 0.$

$\Rightarrow \lim_{a\to\frac{\pi}{4}} I_n(a) = \frac{(-1)^n}{4^n} \pi \cos\frac{a}{8}, \quad n \geq 0.$

$\boxed{\text{Step 3.}}$ Show $N_0(T) \to \infty$ as $T \to \infty$.

Fact: If $f: (a,b) \to \mathbb{R}$ is continuous and never vanishes on (a,b), then

$$\left| \int_a^b f(x)\, dx \right| = \int_a^b |f(x)|\, dx.$$

(The result $N_0(T) > AT$ was proved by counting sign changes of $Z(t) := \frac{\eta(t)}{\frac{1}{2}(\frac{1}{4}+t^2)\pi^{-\frac{1}{4}} |\Gamma(\frac{1}{4}+i\frac{t}{2})|}$.)

Assume $\eta(t)$ has only finitely many zeros. WLOG, assume $\eta(t) > 0$ on (T, ∞) for some $T \geq 2$. Let

$$L := \lim_{a\to\frac{\pi}{4}} \int_T^\infty t^{2n} \cosh(at) \cdot \frac{\eta(t)}{\frac{1}{4}+t^2}\, dt > 0. \quad (\text{Since } \lim_{a\to\frac{\pi}{4}} I_n(a) \text{ exists})$$

So

$$\int_T^{T'} t^{2n}\cosh\left(\frac{\pi t}{4}\right)\cdot\frac{\eta(t)}{\frac{1}{4}+t^2}\,dt \le L \quad , \quad \forall\, T' \ge T.$$

$$\Rightarrow \int_0^{\infty} t^{2n}\cosh\left(\frac{\pi t}{4}\right)\cdot\frac{\eta(t)}{\frac{1}{4}+t^2}\,dt$$ converges absolutely. Thus

$$\int_0^{\infty} t^{2n}\cosh\left(\frac{\pi t}{4}\right)\frac{\eta(t)}{\frac{1}{4}+t^2}\,dt = \lim_{a\to\frac{\pi}{4}} I_n(a) = \frac{(-1)^n}{4^n}\pi\cos\frac{\pi}{8}.$$

For odd $n \ge 1$,

$$\int_0^{\infty} t^{2n}\cosh\left(\frac{\pi t}{4}\right)\frac{\eta(t)}{\frac{1}{4}+t^2}\,dt < 0.$$

$$\|$$

$$\int_0^T + \int_T^{\infty}$$

$$\Rightarrow \int_T^{\infty} t^{2n}\cosh\left(\frac{\pi t}{4}\right)\cdot\frac{\eta(t)}{\frac{1}{4}+t^2}\,dt < \left|\int_0^T t^{2n}\cosh\left(\frac{\pi t}{4}\right)\cdot\frac{\eta(t)}{\frac{1}{4}+t^2}\,dt\right|$$

$$\ll_T \int_0^T t^{2n}\,dt \ll_T T^{2n+1}.$$

But

$$\int_T^{\infty} t^{2n}\cosh\left(\frac{\pi t}{4}\right)\cdot\frac{\eta(t)}{\frac{1}{4}+t^2}\,dt \ge \int_T^{3T} t^{2n}\cosh\left(\frac{\pi t}{4}\right)\cdot\frac{\eta(t)}{\frac{1}{4}+t^2}\,dt \gg_T \int_{2T}^{3T} t^{2n}\,dt \gg_T (2T)^{2n}\,T.$$

Hence $2^{2n} \ll_T 1$. False when n is sufficiently large!